Computing the Angular Velocity Impulse
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First we compute velocities at the collision point on object A and object B (they're dependant on both linear and angular velocity):

$$\bar{\mathbf{v}}_a = \mathbf{v}_a + \boldsymbol{\omega}_a \times \mathbf{r}_a$$
$$\bar{\mathbf{v}}_b = \mathbf{v}_b + \boldsymbol{\omega}_b \times \mathbf{r}_b$$

The terms $\mathbf{r}_a$ and $\mathbf{r}_b$ are vectors to the collision point from the centers of mass of A and B, respectively.

From this, we can compute the relative velocity at the collision point:

$$\bar{\mathbf{v}}_{ab} = \bar{\mathbf{v}}_a - \bar{\mathbf{v}}_b$$

Since momentum is conserved, we can relate incoming and outgoing linear momentum by:

$$m_a \mathbf{v}_a + j\mathbf{n} = m_a \mathbf{v}'_a$$
$$m_b \mathbf{v}_b - j\mathbf{n} = m_b \mathbf{v}'_b$$

I.e. after we apply the impulse along the collision normal, the outgoing momentum is the same. We can divide by the masses to give us the outgoing velocities in terms of the incoming velocities:

$$\mathbf{v}'_a = \mathbf{v}_a + \frac{j}{m_a}\mathbf{n}$$
$$\mathbf{v}'_b = \mathbf{v}_b - \frac{j}{m_b}\mathbf{n}$$

We can do the same thing for angular momentum, and rewrite to get the change in angular velocity:

$$\mathbf{J}_a \boldsymbol{\omega}_a + \mathbf{r}_a \times j\mathbf{n} = \mathbf{J}_a \boldsymbol{\omega}'_a$$
$$\mathbf{J}_b \boldsymbol{\omega}_b - \mathbf{r}_b \times j\mathbf{n} = \mathbf{J}_b \boldsymbol{\omega}'_b$$

or

$$\boldsymbol{\omega}'_a = \boldsymbol{\omega}_a + \mathbf{J}_a^{-1}(\mathbf{r}_a \times j\mathbf{n})$$
$$\boldsymbol{\omega}'_b = \boldsymbol{\omega}_b - \mathbf{J}_b^{-1}(\mathbf{r}_b \times j\mathbf{n})$$

The term **J** is the inertial tensor (I use **J** to distinguish it from the identity matrix).

From all these terms, we can compute the relative velocity *after* the collision:

$$\begin{aligned}\overline{\mathbf{v}}'_{ab} &= \overline{\mathbf{v}}'_a - \overline{\mathbf{v}}'_b \\ &= \mathbf{v}'_a + \boldsymbol{\omega}'_a \times \mathbf{r}_a - \mathbf{v}'_b - \boldsymbol{\omega}'_b \times \mathbf{r}_b \\ &= \left(\mathbf{v}_a + \frac{j}{m_a}\mathbf{n}\right) + \left(\boldsymbol{\omega}_a + \mathbf{J}_a^{-1}(\mathbf{r}_a \times j\mathbf{n})\right) \times \mathbf{r}_a - \left(\mathbf{v}_b - \frac{j}{m_b}\mathbf{n}\right) - \left(\boldsymbol{\omega}_b - \mathbf{J}_b^{-1}(\mathbf{r}_b \times j\mathbf{n})\right) \times \mathbf{r}_b\end{aligned}$$

To solve for j, we need one final equation. The outgoing relative velocity along the collision normal is related to the incoming relative velocity by the coefficient of restitution, or

$$\mathbf{v}'_n = -\varepsilon \mathbf{v}_n$$

The relative velocity along the normal is computed by dotting it with the collision normal, or

$$\overline{\mathbf{v}}'_{ab} \bullet \hat{\mathbf{n}} = -\varepsilon \overline{\mathbf{v}}_{ab} \bullet \hat{\mathbf{n}}$$

Into this we can substitute our equations for $\overline{\mathbf{v}}'_{ab}$ and $\overline{\mathbf{v}}_{ab}$, and then solve for j, to get:

$$j = \frac{-(1+\varepsilon)\overline{\mathbf{v}}_{ab} \bullet \mathbf{n}}{\mathbf{n} \bullet \mathbf{n}\left(\frac{1}{m_a} + \frac{1}{m_b}\right) + \left[\left(\mathbf{J}_a^{-1}(\mathbf{r}_a \times \mathbf{n})\right) \times \mathbf{r}_a + \left(\mathbf{J}_b^{-1}(\mathbf{r}_b \times \mathbf{n})\right) \times \mathbf{r}_b\right] \bullet \mathbf{n}}$$

